

# Covering Numbers of $L_p$ -balls of Convex Sets and Functions

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## Abstract

We prove bounds for the covering numbers of classes of convex functions and convex sets in Euclidean space. Previous results require the underlying convex functions or sets to be uniformly bounded. We relax this assumption and replace it with weaker integral constraints. The existing results can be recovered as special cases of our results.

**Keywords:** covering numbers, packing numbers, convex functions, integral constraints, metric entropy, Komogorov  $\epsilon$ -entropy.

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## 1 Introduction

For a subset  $\mathcal{F}$  of a space  $\mathcal{X}$  equipped with a pseudometric  $\rho$ , the  $\epsilon$ -covering number  $M(\mathcal{F}, \epsilon; \rho)$  is defined as the smallest number of closed balls of radius  $\epsilon$  whose union contains  $\mathcal{F}$ . The quantity  $\log M(\mathcal{F}, \epsilon; \rho)$  is referred to as the  $\epsilon$ -metric entropy. Covering numbers and metric entropy provide an important measure of the massivity of  $\mathcal{F}$  and play a central role in a number of areas including approximation theory, empirical processes, nonparametric function estimation and statistical learning theory.

In this paper, we study the covering numbers of classes of convex functions and classes of convex sets in Euclidean space. For classes of convex functions, the best existing results are due to Dryanov [2] (for  $d = 1$ ) and Guntuboyina and Sen [4] (for  $d \geq 1$ ) who proved optimal upper and lower bounds for the covering numbers of uniformly bounded convex functions under  $L^q$  metrics for  $1 \leq q < \infty$  (the definition of  $L^q$  metrics is recalled in (2)). Specifically, they considered the class  $\mathcal{C}_\infty(I, B)$  of all convex functions

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on  $I := [a_1, b_1] \times \cdots \times [a_d, b_d]$  which are uniformly bounded by  $B$  and proved optimal upper and lower bounds (upto multiplicative constants) for  $\log M(\mathcal{C}_\infty(I, B), \epsilon; L_q(I))$  for  $1 \leq q < \infty$ . These results can be seen as an improvement over the classical results of Bronshtein [1] who considered convex functions that are uniformly Lipschitz in addition to being uniformly bounded.

A natural question regarding the results of Dryanov [2] and Guntuboyina and Sen [4] is whether the uniform boundedness assumption is necessary for obtaining  $L^q$  covering numbers on classes of convex functions. We address this question in this paper and we show that uniform boundedness is not necessary and it can be replaced by an  $L^p$  constraint for any  $p > q$ . Specifically, we consider, for  $1 \leq p < \infty$ , the class  $\mathcal{C}_p(I, B)$  of all convex functions on  $I := [a_1, b_1] \times \cdots \times [a_d, b_d]$  which satisfy the integral constraint  $\int_I |f(x)|^p dx \leq B^p$  and we prove the following interesting phenomenon for  $\log M(\mathcal{C}_p(I, B), \epsilon; L_q(I))$ : for  $1 \leq q < p \leq \infty$ , the metric entropy is finite and is bounded from above and below by constant multiples of  $\epsilon^{-d/2}$  while for  $1 \leq p \leq q \leq \infty$ , the metric entropy is infinite. The results of Dryanov [2] and Guntuboyina and Sen [4] can therefore be seen as special cases of our results corresponding to the case when  $p = \infty$ .

We also prove that, for the case when  $1 \leq p = q < \infty$ , the metric entropy is barely infinite in the following sense: for every subrectangle  $J := [\alpha_1, \beta_1] \times \cdots \times [\alpha_d, \beta_d]$  of  $I$  with  $a_i < \alpha_i < \beta_i < b_i$  for  $i = 1, \dots, d$ , the metric entropy  $\log M(\mathcal{C}_p(I, B), \epsilon; L_p(J))$  is bounded from above by  $\epsilon^{-d/2}$  upto multiplicative factors that are logarithmic in the lengths  $\alpha_i - a_i$  and  $b_i - \beta_i$  for  $i = 1, \dots, d$ .

We also consider classes of convex sets. Here the main existing result on covering numbers is due to Bronshtein [1] who considered the class  $\mathcal{K}_\infty(R)$  of all compact convex subsets of  $\mathbb{R}^d$  (for  $d \geq 2$ ) that are contained in the ball of radius  $R$  centered at the origin. Under the Hausdorff metric  $\ell_H$  (the definition of the Hausdorff metric is recalled in (24)), Bronshtein [1] proved bounds for the metric entropy of  $\mathcal{K}_\infty(R)$ . Specifically, Bronshtein [1] proved that  $\log M(\mathcal{K}_\infty(R), \epsilon; \ell_H)$  is bounded from above and below by constant multiples of  $\epsilon^{(1-d)/2}$ . A similar but weaker result is proved in Dudley [3].

Using the notion of support function, the class  $\mathcal{K}_\infty(R)$  can be thought of as an  $L_\infty$ -ball in the class of all compact, convex subsets of  $\mathbb{R}^d$ . The support function  $h_K$  of a compact, convex subset  $K$  of  $\mathbb{R}^d$  ( $d \geq 2$ ) is defined for  $u$  in the unit sphere,  $S^{d-1} := \{x \in \mathbb{R}^d : x_1^2 + \cdots + x_d^2 = 1\}$ , by

$$h_K(u) := \sup_{x \in K} (x \cdot u) \quad \text{where } x \cdot u := x_1 u_1 + \cdots + x_d u_d.$$

Elementary properties of the support function can be found in [6, Section 1.7] or [5, Section 13]. Using the support function, the class  $\mathcal{K}_\infty(R)$  can be written as  $\{K \in \mathcal{K} : \sup_{u \in S^{d-1}} |h_K(u)| \leq R\}$  where  $\mathcal{K}$  is the class of all compact, convex subsets of  $\mathbb{R}^d$ . A natural question now is to ask for covering numbers of the classes:

$$\mathcal{K}_p(R) := \left\{ K \in \mathcal{K} : \int_{S^{d-1}} |h_K(u)|^p d\nu(u) \leq R^p \right\} \quad \text{for } 1 \leq p < \infty \quad (1)$$

where  $\nu$  is the uniform probability measure on  $S^{d-1}$ . These classes are all larger than  $\mathcal{K}_\infty(R)$ . In Theorem 5.2 of this paper, we prove that, for every  $1 \leq p \leq \infty$ , the metric entropy  $\log M(\mathcal{K}_p(R), \epsilon; \ell_H)$  is bounded from above and below by constant multiples of  $\epsilon^{(1-d)/2}$ .

The rest of the paper is organized as follows. We state our results for the metric entropy of classes of convex functions in Section 2. We prove these results in Section 4. The main idea behind our convex function results can be isolated into a separate theorem which we state and prove in Section 3. Our results for convex sets are stated and proved in Section 5. The proof of an auxiliary result is given in Section 6.

## 2 Convex Functions

Recall the notions of  $\mathcal{C}_p(I, B)$  for  $1 \leq p \leq \infty$ ,  $I = [a_1, b_1] \times \cdots \times [a_d, b_d]$  and  $B > 0$ . Also recall that under the  $L_q(J)$  metric on a subset  $J$  of  $\mathbb{R}^d$ , the distance between two functions  $f$  and  $g$  on  $J$  is defined as

$$\left( \int_J |f(x) - g(x)|^q dx \right)^{1/q} \quad \text{for } 1 \leq q < \infty \quad (2)$$

and as  $\sup_{x \in J} |f(x) - g(x)|$  for  $q = \infty$ .

Guntuboyina and Sen [4, Theorem 3.1] proved the following result for the metric entropy of  $\mathcal{C}_\infty(I, B)$  under the  $L_q(I)$  metric for  $1 \leq q < \infty$ . Dryanov [2] previously proved the special case of this result for  $d = 1$ .

**Theorem 2.1** (Guntuboyina and Sen). *Fix  $d \geq 1$  and  $1 \leq q < \infty$ . There exist positive constants  $c_1, c_2$  and  $\epsilon_0$  depending only on  $d$  and  $q$  such that for every  $B > 0$  and  $I = [a_1, b_1] \times \cdots \times [a_d, b_d]$ , we have*

$$\log M(\mathcal{C}_\infty(I, B), \epsilon; L_q(I)) \leq c_1 \left( \frac{\epsilon}{B(b_1 - a_1)^{1/q} \cdots (b_d - a_d)^{1/q}} \right)^{-d/2} \quad (3)$$

for all  $\epsilon > 0$  and

$$\log M(\mathcal{C}_\infty(I, B), \epsilon; L_q(I)) \geq c_2 \left( \frac{\epsilon}{B(b_1 - a_1)^{1/q} \cdots (b_d - a_d)^{1/q}} \right)^{-d/2} \quad (4)$$

whenever  $0 < \epsilon \leq \epsilon_0 B(b_1 - a_1)^{1/q} \cdots (b_d - a_d)^{1/q}$ .

**Remark 2.1.** In [4], the above result was proved only for rectangles of the form  $[a, b]^d$  as opposed to  $[a_1, b_1] \times \cdots \times [a_d, b_d]$ . But it is easy to see that the result for  $[a, b]^d$  implies Theorem 2.1 by a scaling argument (for example, via inequality (14)).

**Remark 2.2.** In [4], inequality (3) was only proved for  $\epsilon \leq \epsilon_0 B(b_1 - a_1)^{1/q} \cdots (b_d - a_d)^{1/q}$  for a positive constant  $\epsilon_0$  depending only on  $d$  and  $q$ . It turns out however that this condition is redundant. This follows from the observation that the diameter of the space  $\mathcal{C}_\infty(I, B)$  in the  $L_q(I)$  metric is at most  $2B(b_1 - a_1)^{1/q} \cdots (b_d - a_d)^{1/q}$  which means that left hand side of (3) equals 0 for  $\epsilon > 2B(b_1 - a_1)^{1/q} \cdots (b_d - a_d)^{1/q}$ .

In this paper, we extend Theorem 2.1 by proving bounds for the metric entropy of  $\mathcal{C}_p(I, B)$  for  $1 \leq p < \infty$ . Note that functions in  $\mathcal{C}_p(I, B)$  for  $1 \leq p < \infty$  do not have to be uniformly bounded as

in  $\mathcal{C}_\infty(I, B)$  but instead they are only required to satisfy a weaker integral constraint. We prove the following result for the metric entropy of these classes under  $L_q$  metrics: for  $q < p$ , the metric entropy under the  $L_q$  metric is finite and is bounded from above by a constant multiple of  $\epsilon^{-d/2}$  while for  $q \geq p$ , the metric entropy under the  $L_q$  metric is infinite. The fact that the metric entropy is infinite when  $q \geq p$  is shown in Theorem 2.3 while bounds on the metric entropy for  $q < p$  are proved in Theorem 2.2. It is clear that Theorem 2.1 is a special case of Theorem 2.2 corresponding to  $p = \infty$ .

**Theorem 2.2.** *Fix  $d \geq 1$  and  $1 \leq q < p \leq \infty$ . There exist positive constants  $c_1, c_2$  and  $\epsilon_0$  depending only on  $d, p$  and  $q$  such that*

$$\log M(\mathcal{C}_p(I, B), \epsilon; L_q) \leq c_1 \left( \frac{\epsilon}{B(b_1 - a_1)^{1/q-1/p} \dots (b_d - a_d)^{1/q-1/p}} \right)^{-d/2} \quad (5)$$

for every  $\epsilon > 0$  and

$$\log M(\mathcal{C}_p(I, B), \epsilon; L_q) \geq c_2 \left( \frac{\epsilon}{B(b_1 - a_1)^{1/q-1/p} \dots (b_d - a_d)^{1/q-1/p}} \right)^{-d/2} \quad (6)$$

whenever  $0 < \epsilon \leq \epsilon_0 B(b_1 - a_1)^{1/q-1/p} \dots (b_d - a_d)^{1/q-1/p}$ .

**Theorem 2.3.** *Fix  $d \geq 1$  and  $1 \leq p \leq q \leq \infty$ . There exists a positive constant  $\epsilon_0$  depending only on  $d, p$  and  $q$  such that*

$$\log M(\mathcal{C}_p(I, B), \epsilon; L_q(I)) = \infty \quad (7)$$

whenever  $\epsilon \leq B\epsilon_0(b_1 - a_1)^{1/q-1/p} \dots (b_d - a_d)^{1/q-1/p}$ .

When  $1 \leq p = q < \infty$ , it turns out that  $\log M(\mathcal{C}_p(I, B), \epsilon; L_p(I))$  is barely infinite in the sense made precise by the theorem below. Note that the dependence on  $\eta$  in the next theorem is logarithmic.

**Theorem 2.4.** *Fix  $d \geq 1$  and  $1 \leq p < \infty$ . There exists a positive constant  $c$  depending only  $d$  and  $p$  such that for every  $I = [a_1, b_1] \times \dots \times [a_d, b_d]$  and  $J = [\alpha_1, \beta_1] \times \dots \times [\alpha_d, \beta_d]$  with  $a_i < \alpha_i < \beta_i < b_i$  with  $i = 1, \dots, d$ , we have*

$$\log M(\mathcal{C}_p(I, B), \epsilon; L_p(J)) \leq c \left( \frac{\epsilon}{B} \right)^{-d/2} \left( \log \frac{1}{2\eta} \right)^{d(2p+d)/(2p)}$$

for all  $\epsilon > 0$  where

$$\eta := \min \left( \frac{\alpha_1 - a_1}{b_1 - a_1}, \dots, \frac{\alpha_d - a_d}{b_d - a_d}, \frac{b_1 - \beta_1}{b_1 - a_1}, \dots, \frac{b_d - \beta_d}{b_d - a_d} \right). \quad (8)$$

The main idea behind the proofs of Theorem 2.2 and 2.4 is the following: scaling identities (14) and (15) described in Section 4 allow us to take  $I = [0, 1]^d$ . We then show that functions in  $\mathcal{C}_p([0, 1]^d, 1)$  are uniformly bounded on subrectangles that are contained in the interior of  $[0, 1]^d$ . We divide  $[0, 1]^d$  into such subrectangles and apply Theorem 2.1 in each of the subrectangles. The proofs are then completed by combining the metric entropy bounds from Theorem 2.1 for each of the subrectangles. This method, which we call the partitioning method, can be isolated into a separate theorem (Theorem 3.1) which we state and prove in the next section. We then show how Theorems 2.2 and 2.4 can be proved from Theorem 3.1 in Section 4 where we also provide the proof of Theorem 2.3.

### 3 The Partitioning Theorem

**Theorem 3.1.** *Fix  $1 \leq p < \infty$  and  $1 \leq q < \infty$ . There exists a constant  $c$  depending only on  $d$ ,  $p$  and  $q$  such that the following inequality is true for every  $0 < \eta < u \leq 1/2$ ,  $l \geq 1$  and every finite sequence  $\eta = \eta_0 < \eta_1 < \dots < \eta_l < u \leq \eta_{l+1}$ :*

$$\log M(\mathcal{C}_p([0, 1]^d, 1), \epsilon, L_q[\eta, u]^d) \leq c\epsilon^{-d/2} \left( \sum_{i=0}^l \frac{(\eta_{i+1} - \eta_i)^{d/(2q+d)}}{\eta_i^{dq/(p(2q+d))}} \right)^{d(2q+d)/(2q)} \quad \text{for all } \epsilon > 0. \quad (9)$$

We need two preparatory results for the proof of Theorem 3.1. The first of these results is given below. Its proof is trivial and is omitted.

**Lemma 3.2.** *Let  $\mathcal{F}$  be an arbitrary class of functions defined on a subset  $A$  of  $\mathbb{R}^d$  and let  $A_1, \dots, A_k$  denote subsets of  $\mathbb{R}^d$  with  $A \subseteq \cup_{i=1}^k A_i$ . Then for every  $\epsilon, \epsilon_1, \dots, \epsilon_k > 0$ , we have*

$$\log M(\mathcal{F}, \epsilon; L_q(A)) \leq \sum_{i=1}^k \log M(\mathcal{F}, \epsilon_i, L_q(A_i)) \quad \text{provided } \sum_{i=1}^k \epsilon_i^q \leq \epsilon^q.$$

The second preparatory result states that for every  $\phi \in \mathcal{C}_p([0, 1]^d, 1)$  and  $y \in (0, 1)^d$ , the quantity  $|\phi(y)|$  can be bounded from above by a term that is independent of  $\phi$ . The precise statement is given below and its proof is deferred to Section 6. .

**Lemma 3.3.** *Let  $1 \leq p \leq \infty$  and let  $\phi$  be a convex function on  $[0, 1]^d$  with  $\int_{[0, 1]^d} |\phi(x)|^p dx \leq 1$ . Then there exists a positive constant  $c$  depending only on  $d$  and  $p$  such that for every  $y = (y_1, \dots, y_d) \in (0, 1)^d$ ,*

$$|\phi(y)| \leq c \prod_{i=1}^d \max \left( y_i^{-1/p}, (1 - y_i)^{-1/p} \right). \quad (10)$$

We are now ready to prove Theorem 3.1.

*Proof of Theorem 3.1.* Let us fix  $0 < \eta < u \leq 1/2$  and an arbitrary finite sequence  $\eta = \eta_0 < \eta_1 < \dots < \eta_l < u \leq \eta_{l+1}$  for a positive integer  $l \geq 1$ . For every  $f$  and  $g$ , we can write

$$\int_{[\eta, u]^d} |f(x) - g(x)|^q dx \leq \sum_{i_1=0}^l \dots \sum_{i_d=0}^l \int_{\eta_{i_1}}^{\eta_{i_1+1}} \dots \int_{\eta_{i_d}}^{\eta_{i_d+1}} |f(x) - g(x)|^q dx_1 \dots dx_d.$$

Lemma 3.3 asserts that every function  $\phi \in \mathcal{C}_p([0, 1]^d, 1)$ , when restricted to the rectangle  $[\eta_{i_1}, \eta_{i_1+1}] \times \dots \times [\eta_{i_d}, \eta_{i_d+1}]$ , is convex and uniformly bounded by  $C\eta_{i_1}^{-1/p} \dots \eta_{i_d}^{-1/p}$  for a constant  $C$  that only depends on  $d$  and  $p$ . Therefore, by Theorem 2.1, we can cover the restrictions of functions in  $\mathcal{C}_p([0, 1]^d, 1)$  to  $[\eta_{i_1}, \eta_{i_1+1}] \times \dots \times [\eta_{i_d}, \eta_{i_d+1}]$  to within a positive real number  $\alpha(i_1, \dots, i_d)$  in the  $L_q$  metric on  $[\eta_{i_1}, \eta_{i_1+1}] \times \dots \times [\eta_{i_d}, \eta_{i_d+1}]$  by a finite set having cardinality atmost

$$\exp \left( c \left( \frac{\alpha(i_1, \dots, i_d) \eta_{i_1}^{1/p} \dots \eta_{i_d}^{1/p}}{(\eta_{i_1+1} - \eta_{i_1})^{1/q} \dots (\eta_{i_d+1} - \eta_{i_d})^{1/q}} \right)^{-d/2} \right)$$

where  $c$  is a positive constant that only depends on  $d$ ,  $p$  and  $q$ . By Lemma 3.2 therefore, we get an  $\epsilon$ -cover for functions in  $\mathcal{C}_p([0, 1]^d, 1)$  in the  $L_q$  metric on  $[\eta, u]^d$  with

$$\epsilon^q = \sum_{i_1=0}^l \cdots \sum_{i_d=0}^l \alpha^q(i_1, \dots, i_d)$$

having cardinality at most

$$\exp \left[ c \sum_{i_1=0}^l \cdots \sum_{i_d=0}^l \left( \frac{\alpha(i_1, \dots, i_d) \eta_{i_1}^{1/p} \cdots \eta_{i_d}^{1/p}}{(\eta_{i_1+1} - \eta_{i_1})^{1/q} \cdots (\eta_{i_d+1} - \eta_{i_d})^{1/q}} \right)^{-d/2} \right]. \quad (11)$$

For each  $i := (i_1, \dots, i_d) \in \{0, \dots, l\}^d$ , let

$$u_i := \frac{\eta_{i_1}^{1/p} \cdots \eta_{i_d}^{1/p}}{(\eta_{i_1+1} - \eta_{i_1})^{1/q} \cdots (\eta_{i_d+1} - \eta_{i_d})^{1/q}}. \quad (12)$$

Plugging in the choice

$$\alpha(i_1, \dots, i_d) := \epsilon u_i^{-d/(d+2q)} \left( \sum_{i \in \{0, \dots, l\}^d} u_i^{-dq/(d+2q)} \right)^{-1/q}$$

into (11), we obtain that

$$\log M(\epsilon, \mathcal{C}_p([0, 1]^d, 1), L_q[\eta, u]^d) \leq c \epsilon^{-d/2} \left( \sum_i u_i^{-dq/(2q+d)} \right)^{(2q+d)/(2q)}. \quad (13)$$

The observation

$$\sum_{i \in \{0, \dots, l\}^d} u_i^{-dq/(2q+d)} = \left( \sum_{i=0}^l \frac{(\eta_{i+1} - \eta_i)^{d/(2q+d)}}{\eta_i^{dq/(p(2q+d))}} \right)^d.$$

now completes the proof.  $\square$

## 4 Proofs for results in Section 2

We give the proofs of Theorems 2.2, 2.3 and 2.4 in this section. We start with a pair of simple scaling identities which allow us to take  $I = [0, 1]^d$  without loss of generality. The first identity is: For every  $I = [a_1, b_1] \times \cdots \times [a_d, b_d]$ , we have

$$M(\mathcal{C}_p(I, B), \epsilon; L_q(I)) = M(\mathcal{C}_p([0, 1]^d, 1), \tilde{\epsilon}, L_q([0, 1]^d)) \quad (14)$$

where

$$\tilde{\epsilon} := (b_1 - a_1)^{1/p-1/q} \cdots (b_d - a_d)^{1/p-1/q} \frac{\epsilon}{B}.$$

To see (14), associate for each  $f \in \mathcal{C}_p(I, B)$ , the function  $\tilde{f}$  on  $[0, 1]^d$  by

$$\tilde{f}(x_1, \dots, x_d) := B^{-1} (b_1 - a_1)^{1/p} \cdots (b_d - a_d)^{1/p} f(a_1 + (b_1 - a_1)x_1, \dots, a_d + (b_d - a_d)x_d).$$

It is then easy to verify that  $\tilde{f}$  lies in  $\mathcal{C}_p([0, 1]^d, 1)$  and that covering  $\tilde{f}$  to within  $\tilde{\epsilon}$  in the  $L_q$  metric on  $[0, 1]^d$  is equivalent to covering  $f$  to within  $\epsilon$  in the  $L_q$  metric on  $I$  and this proves (14). The identity (14) implies that we can, without loss of generality, take  $I = [0, 1]^d$  and  $B = 1$  in the proofs of Theorems 2.2 and 2.3.

The second scaling identity is: For every  $I = [a_1, b_1] \times \cdots \times [a_d, b_d]$  and  $J = [\alpha_1, \beta_1] \times \cdots \times [\alpha_d, \beta_d]$  with  $a_i < \alpha_i < \beta_i < b_i$  for all  $i$ , we have

$$M(\mathcal{C}_p(I, B), \epsilon, L_p(I)) \leq M(\mathcal{C}_p([0, 1]^d, 1), \epsilon/B, L_p[\eta, 1 - \eta]^d) \quad (15)$$

where  $\eta$  is defined as in (8). The proof of (15) is similar to that of (14) and is thus omitted. Identity (15) allows us to take, without loss of generality,  $I = [0, 1]^d$ ,  $J = [\eta, 1 - \eta]^d$  and  $B = 1$  for the proof of Theorem 2.4.

## 4.1 Proof of Theorem 2.2

Inequality (6) is a direct consequence of (4) because

$$\mathcal{C}_\infty(I, B(b_1 - a_1)^{-1/p} \cdots (b_d - a_d)^{-1/p}) \subseteq \mathcal{C}_p(I, B) \quad \text{for every } 1 \leq p \leq \infty.$$

We therefore only need to prove (5). We assume that  $p < \infty$  because the case when  $p = \infty$  is taken care of by Theorem 2.1. The scaling inequality (14) allows us to restrict attention to the case when  $I = [0, 1]^d$  and  $B = 1$ . Therefore, we only need to prove the existence of a positive constant  $c$  (depending only on  $d, p$  and  $q$ ) such that

$$\log M(\mathcal{C}_p([0, 1]^d, 1), \epsilon; L_q[0, 1]^d) \leq c\epsilon^{-d/2} \quad \text{for all } \epsilon > 0. \quad (16)$$

Our first step for the proof of (16) is to reduce focus to the  $L_q$  metric on a subrectangle  $[\eta, 1/2]^d$  of  $[0, 1]^d$  for some  $\eta > 0$  as opposed to the  $L_q$  metric on the entire unit cube  $[0, 1]^d$ .

### 4.1.1 Reduction to the $L_q[\eta, 1/2]^d$ metric for $0 < \eta < 1/2$

The behaviour of functions in  $\mathcal{C}_p([0, 1]^d, 1)$  can be difficult to control near the boundary of the cube  $[0, 1]^d$ . For this reason, the metric entropy of  $\mathcal{C}_p([0, 1]^d, 1)$  under the pseudo-metric  $L_q[\eta, 1 - \eta]^d$  for  $0 < \eta < 1/2$  will be easier to bound than the metric entropy under  $L_q[0, 1]^d$ . The following lemma proves that it is actually enough to work with  $L_q[\eta, 1 - \eta]^d$  for some  $0 < \eta < 1/2$ .

**Lemma 4.1.** *Let*

$$0 < \epsilon < 2^{1/q} d^{(1/q) - (1/p)} \quad \text{and} \quad \eta_\epsilon := \frac{1}{2d} \left( \frac{\epsilon^q}{2} \right)^{p/(p-q)}.$$

*Then*

$$M(\mathcal{C}_p([0, 1]^d, 1), \epsilon; L_q[0, 1]^d) \leq M\left(\mathcal{C}_p([0, 1]^d, 1), \epsilon 2^{-1/q}; L_q[\eta_\epsilon, 1 - \eta_\epsilon]^d\right). \quad (17)$$

*Proof.* Fix  $\phi \in \mathcal{C}_p([0, 1]^d, 1)$  and an arbitrary function  $\psi$  on  $[\eta_\epsilon, 1 - \eta_\epsilon]^d$  where  $\eta_\epsilon$  is defined as in the statement of the lemma. Extend  $\psi$  to  $[0, 1]^d$  by defining it to be zero outside  $[\eta_\epsilon, 1 - \eta_\epsilon]^d$ . Observe that

$$\int_{[0, 1]^d} |\phi - \psi|^q = \int_{[\eta_\epsilon, 1 - \eta_\epsilon]^d} |\phi - \psi|^q + \int_{[0, 1]^d} |\phi(x)|^q I\{x \notin [\eta_\epsilon, 1 - \eta_\epsilon]^d\} dx. \quad (18)$$

Applying Holder's inequality  $\int |fg| \leq (\int |f|^r)^{1/r} (\int |g|^s)^{1/s}$  with  $f := |\phi|^q$ ,  $g := I\{x \notin [\eta_\epsilon, 1 - \eta_\epsilon]^d\}$ ,  $r = p/q$  and  $s = p/(p - q)$ , we get

$$\begin{aligned} \int_{[0, 1]^d} |\phi(x)|^q I\{x \notin [\eta_\epsilon, 1 - \eta_\epsilon]^d\} dx &\leq \left( \int_{[0, 1]^d} |\phi(x)|^p \right)^{q/p} (1 - (1 - 2\eta_\epsilon)^d)^{1-(q/p)} \\ &\leq \left( \int_{[0, 1]^d} |\phi(x)|^p \right)^{q/p} (2d\eta_\epsilon)^{1-(q/p)} \\ &= \frac{\epsilon^q}{2} \left( \int_{[0, 1]^d} |\phi(x)|^p \right)^{q/p} (2d\eta_\epsilon)^{1-(q/p)} \leq \frac{\epsilon^q}{2}. \end{aligned}$$

This, together with (18), gives

$$\int_{[0, 1]^d} |\phi - \psi|^q \leq \int_{[\eta_\epsilon, 1 - \eta_\epsilon]^d} |\phi - \psi|^q + \frac{\epsilon^q}{2}$$

from which (17) follows immediately.  $\square$

By symmetry, it can be shown that the metric entropy of  $\mathcal{C}_p([0, 1]^d, 1)$  under  $L_q[\eta, 1 - \eta]^d$  is bounded from above by a constant multiple of the metric entropy under  $L_q[\eta, 1/2]^d$ . This is the content of the following lemma.

**Lemma 4.2.** *The following inequality holds for every  $0 < \eta < 1/2$*

$$\log M(\mathcal{C}_p([0, 1]^d, 1), \epsilon; L_q[\eta, 1 - \eta]^d) \leq 2^d \log M(\mathcal{C}_p([0, 1]^d, 1), \epsilon 2^{-d/q}, L_q[\eta, 1/2]^d). \quad (19)$$

*Proof.* Let  $I(0) := [\eta, 1/2]$  and  $I(1) := [1/2, 1 - \eta]$ . For any pair of functions  $\phi$  and  $\psi$ , observe that

$$\int_{[\eta, 1 - \eta]^d} |\phi - \psi|^q = \sum_{\theta \in \{0, 1\}^d} \int_{I(\theta_1) \times \cdots \times I(\theta_d)} |\phi - \psi|^q$$

which implies, by Lemma 3.2, that

$$M(\mathcal{C}_p([0, 1]^d, 1), \epsilon; L_q[\eta, 1 - \eta]^d) \leq \prod_{\theta \in \{0, 1\}^d} M(\mathcal{C}_p([0, 1]^d, 1), \epsilon 2^{-d/q}; L_q(I(\theta_1) \times \cdots \times I(\theta_d))).$$

By symmetry, we get that

$$M(\mathcal{C}_p([0, 1]^d, 1), \epsilon 2^{-d/q}; L_q(I(\theta_1) \times \cdots \times I(\theta_d))) = M(\mathcal{C}_p([0, 1]^d, 1), \epsilon 2^{-d/q}, L_q[\eta, 1/2]^d)$$

for every  $\theta \in \{0, 1\}^d$ . This completes the proof of (19).  $\square$

The above pair of results (Lemma 4.1 and 4.2) together imply that (16) will be a consequence of the following result:



**Proposition 4.3.** Fix  $d \geq 1$  and  $1 \leq q < p < \infty$ . There exists positive constants  $c$  and  $\epsilon_0$  depending only on  $d, p$  and  $q$  such that

$$\sup_{0 < \eta < 1/2} \log M(\mathcal{C}_p([0, 1]^d, 1), \epsilon; L_q[\eta, 1/2]^d) \leq c\epsilon^{-d/2} \quad \text{for every } \epsilon \leq \epsilon_0.$$

Proposition 4.3 will be proved in the next subsection. This will complete the proof of Theorem 2.2 .

#### 4.1.2 Proof of Proposition 4.3

Fix  $p > q$  and let

$$u := \exp \left( \frac{-2p(p+q)(2q+d) \log 2}{d(p-q)^2} \right). \quad (20)$$

Note that  $u$  only depends on  $p, q$  and  $d$  and that  $0 < u < 1/2$ .

Using the notation  $a \vee b := \max(a, b)$ , we can write

$$\int_{[\eta, 1/2]^d} |f - g|^q dx = \int_{[\eta, u \vee \eta]^d} |f - g|^q + \int_{[u \vee \eta, 1/2]^d} |f - g|^q \leq \int_{[\eta, u \vee \eta]^d} |f - g|^q + \int_{[u, 1/2]^d} |f - g|^q.$$

for every pair of functions  $f$  and  $g$ . Applying Lemma 3.2, we obtain

$$M(\mathcal{C}_p([0, 1]^d, 1), \epsilon; L_q[\eta, 1/2]^d) \leq M(\mathcal{C}_p([0, 1]^d, 1), 2^{-1/q}\epsilon; L_q[\eta, u \vee \eta]^d) M(\mathcal{C}_p([0, 1]^d, 1), 2^{-1/q}\epsilon; L_q[u, 1/2]^d).$$

Now, by Lemma 3.3, every function in  $\mathcal{C}_p([0, 1]^d, 1)$ , when restricted to  $[u, 1/2]^d$  is convex and uniformly bounded by  $Cu^{-d/p}$  for a positive constant  $C$  that only depends on  $d$  and  $p$ . It follows from Theorem 2.1 (and the fact that  $u$  is a constant that only depends on  $d, p$  and  $q$ ) that there exists a constant  $c$  depending on  $d, p$  and  $q$  alone such that

$$\log M(\mathcal{C}_p([0, 1]^d, 1), 2^{-1/q}\epsilon; L_q[u, 1/2]^d) \leq c\epsilon^{-d/2} \quad \text{for all } \epsilon > 0.$$

We deduce therefore that the proof of Proposition 4.3 will be complete if we prove the existence of a constant  $c$  for which

$$\log M(\mathcal{C}_p([0, 1]^d, 1), \epsilon; L_q[\eta, u \vee \eta]^d) \leq c\epsilon^{-d/2}. \quad (21)$$

We prove (21) below. It is trivial when  $\eta \geq u$  so we assume below that  $\eta < u$ . By Theorem 3.1, there exists a positive constant  $c$  depending only on  $d, p$  and  $q$  such that

$$\log M(\epsilon, \mathcal{C}_p([0, 1]^d, 1), L_q[\eta, u]^d) \leq c\epsilon^{-d/2} \left( \sum_{i=0}^l \frac{(\eta_{i+1} - \eta_i)^{d/(2q+d)}}{\eta_i^{dq/(p(2q+d))}} \right)^{d(2q+d)/(2q)} \quad (22)$$

We use this with

$$\eta_i := \exp \left( \left( \frac{p+q}{2p} \right)^i \log \eta \right) \quad \text{for } i \geq 1$$

and  $l$  taken to be the largest integer  $i$  for which  $\eta_i < u$ . Because  $p > q$  and  $\log \eta < 0$ , it is clear that  $\{\eta_i\}$  is an increasing sequence.

We shall show below that for this choice of  $l$  and  $\{\eta_i\}$ ,

$$S := \sum_{i=0}^l \frac{(\eta_{i+1} - \eta_i)^{d/(2q+d)}}{\eta_i^{dq/(p(2q+d))}} \leq C$$

for a positive constant  $C$  that only depends on  $d$ ,  $p$  and  $q$ . The proof would then be complete by (22).

Define

$$\zeta_i := \frac{\eta_{i+1}^{d/(2q+d)}}{\eta_i^{dq/(p(2q+d))}} = \exp \left( \frac{d(p-q)}{2p(2q+d)} \left( \frac{p+q}{2p} \right)^i \log \eta \right).$$

Observe that for  $1 \leq i \leq l$

$$\begin{aligned} \frac{\zeta_i}{\zeta_{i-1}} &= \exp \left( \frac{-d(p-q)^2}{4p^2(2q+d)} \left( \frac{p+q}{2p} \right)^{i-1} \log \eta \right) \\ &= \exp \left( \frac{-d(p-q)^2}{2p(p+q)(2q+d)} \log \eta_i \right) \geq \exp \left( \frac{-d(p-q)^2}{2p(p+q)(2q+d)} \log u \right) = 2 \end{aligned}$$

where we have used  $\eta_i < u$  for  $1 \leq i \leq l$  and the expression (20) for  $u$ . This means that  $\zeta_i \leq 2(\zeta_i - \zeta_{i-1})$  for  $i = 1, \dots, l$  and, as a result, we get

$$S \leq \sum_{i=0}^l \zeta_i = \zeta_0 + 2 \sum_{i=1}^l (\zeta_i - \zeta_{i-1}) = 2\zeta_l - \zeta_0 \leq 2\zeta_l.$$

$\zeta_l$  can be bounded by a constant independent of  $\eta$  because

$$\zeta_l = \exp \left( \frac{d(p-q)}{2p(2q+d)} \left( \frac{p+q}{2p} \right)^l \log \eta \right) = \exp \left( \frac{d(p-q)}{2p(2q+d)} \log \eta_l \right) < \exp \left( \frac{d(p-q)}{2p(2q+d)} \log u \right).$$

This proves that  $S$  is bounded from above by a constant that only depends on  $d$ ,  $p$  and  $q$ . This completes the proof of Proposition 4.3 and thereby that of Theorem 2.2.

## 4.2 Proof of Theorem 2.3

Because of (14), it is sufficient to prove the theorem for  $a_1 = \dots = a_d = 0$ ,  $b_1 = \dots = b_d = 1$  and  $B = 1$ .

Define, for  $j \geq 1$ ,

$$f_j(x) := (1+p)^{1/p} 2^{j/p} \max(0, 1 - 2^j x_1) \quad \text{for } x = (x_1, \dots, x_d) \in [0, 1]^d.$$

It is easy to check that  $f_j \in \mathcal{C}_p([0, 1]^d, 1)$  for every  $j \geq 1$ . Now for  $j < k$ , note that

$$(1+p)^{-q/p} \int_{[0, 1]^d} |f_j(x) - f_k(x)|^q dx \geq \int_{2^{-k}}^{2^{-j}} 2^{jq/p} (1 - 2^j x_1)^q dx_1 = \frac{2^{j(q-p)/p}}{q+1} (1 - 2^{j-k})^{q+1} \geq c$$

for some positive constant  $c$  that depends only on  $d, p$  and  $q$ . We have thus shown that for every pair of distinct functions from the infinite sequence  $\{f_j\}$ , the  $L_q$  distance between them is bounded from below by a positive constant that only depends on  $p$  and  $q$ . This proves the theorem.

### 4.3 Proof of Theorem 2.4

By the second scaling identity (15), we can take  $I = [0, 1]^d$ ,  $J = [\eta, 1 - \eta]^d$  and  $B = 1$  without loss of generality. Further, by Lemma 4.2, it is enough to bound  $\log M(\mathcal{C}_p([0, 1]^d, 1), \epsilon, L_p[\eta, 1/2]^d)$ .

Theorem 3.1 with  $p = q$  and  $u = 1/2$  gives the existence of a constant  $c$  for which

$$\log M(\mathcal{C}_p([0, 1]^d, 1), \epsilon, L_p[\eta, 1/2]^d) \leq c\epsilon^{-d/2} \left( \sum_{i=0}^l \frac{(\eta_{i+1} - \eta_i)^{d/(2p+d)}}{\eta_i^{d/(2p+d)}} \right)^{d(2p+d)/(2p)} \quad (23)$$

for every  $l \geq 1$  and every  $\eta = \eta_0 < \eta_1 < \dots < \eta_l < 1/2 \leq \eta_{l+1}$ . We apply this to  $\eta_i = 2^{-i}\eta$  for  $i = 0, \dots, l$  where  $l := \lfloor -\log(2\eta)/\log 2 \rfloor$  and  $\eta_{l+1} = 1/2$ . Here  $\lfloor x \rfloor$  is the largest integer that is strictly smaller than  $x$ . It is clear then that  $\eta_{i+1} - \eta_i \leq \eta_i$  and then, using (23), we get

$$\log M(\mathcal{C}_p([0, 1]^d, 1), \epsilon, L_p[\eta, 1/2]^d) \leq c\epsilon^{-d/2} (l+1)^{d(2p+d)/(2p)} \leq c_1\epsilon^{-d/2} \left( \log \frac{1}{2\eta} \right)^{d(2p+d)/(2p)}$$

where  $c_1$  only depends on  $d, p$  and  $q$ . This completes the proof of Theorem 2.4.

## 5 Convex sets

Recall the definition of the classes  $\mathcal{K}_p(R)$  for  $1 \leq p \leq \infty$  and  $d \geq 1$  from (1). Bronshtein [1, Theorem 3 and Remark 1] proved the following bound on the covering number of  $\mathcal{K}_\infty(R)$  under the Hausdorff metric  $\ell_H$ . It may be recalled that the Hausdorff distance between two compact, convex sets  $C$  and  $D$  in Euclidean space is defined by

$$\ell_H(C, D) := \max \left( \sup_{x \in C} \inf_{y \in D} |x - y|, \sup_{x \in D} \inf_{y \in C} |x - y| \right) \quad (24)$$

where  $|\cdot|$  denotes Euclidean distance.

**Theorem 5.1** (Bronshtein). *There exist positive constants  $c_1$  and  $c_2$  depending only on  $d$  such that for every  $R > 0$ ,*

$$c_1 \left( \frac{R}{\epsilon} \right)^{(d-1)/2} \leq \log M(\mathcal{K}_\infty(R), \epsilon; \ell_H) \leq c_2 \left( \frac{R}{\epsilon} \right)^{(d-1)/2}. \quad (25)$$

In the next theorem, we show that the same result (25) holds for the covering number of  $\mathcal{K}_p(R)$  for every  $1 \leq p \leq \infty$ . To the best of our knowledge, this result is new. Note that the classes  $\mathcal{K}_p(R)$  for  $1 \leq p < \infty$  are all larger than  $\mathcal{K}_\infty(R)$ .

**Theorem 5.2.** *There exist positive constants  $c_1$  and  $c_2$  depending only on  $d$  such that for every  $1 \leq p \leq \infty$  and  $R > 0$ ,*

$$c_1 \left( \frac{R}{\epsilon} \right)^{(d-1)/2} \leq \log M(\mathcal{K}_p(R), \epsilon; \ell_H) \leq c_2 \left( \frac{R}{\epsilon} \right)^{(d-1)/2}. \quad (26)$$

*Proof.*  $p = \infty$  corresponds to Theorem 5.1 so we may assume that  $1 \leq p < \infty$ . Because  $\mathcal{K}_\infty(R) \subseteq \mathcal{K}_p(R)$ , the lower bound on  $M(\mathcal{K}_p(R), \epsilon; \ell_H)$  follows from Theorem 5.1. We therefore only need to prove the upper bound. We show below that there exists a positive constant  $M$  depending only on  $d$  and  $p$  such that

$$\mathcal{K}_p(R) \subseteq \mathcal{K}_\infty(MR) \quad (27)$$

This means that  $M(\mathcal{K}_p(R), \epsilon; \ell_H) \leq M(\mathcal{K}_\infty(MR), \epsilon; \ell_H)$ . The proof will then be complete by the use of Theorem 5.1.

For each  $v \in S^{d-1}$ , define the spherical cap  $\mathcal{S}(v) := \{x \in S^{d-1} : \|x - v\|^2 \leq 1\}$ . It is easy to check that  $\mathcal{S}(v)$  can also be written as  $\{x \in S^{d-1} : x \cdot v \geq 1/2\}$ .

To prove (27), fix  $K \in \mathcal{K}_p(R)$  and  $x \in K$ . We need to show that  $x \in MR$  for a constant  $M$  which only depends on  $d$  and  $p$ . We may clearly assume that  $x \neq 0$  and let  $v := x/\|x\|$ . Note that for every  $u \in S^{d-1}$ , we have  $h_K(u) \geq x \cdot u = \|x\|(v \cdot u)$ . Consequently,  $h_K(u) \geq \|x\|/2$  whenever  $u \in \mathcal{S}(v)$ . As a result,

$$R^p \geq \int_{S^{d-1}} |h_K(u)|^p d\nu(u) \geq \int_{\mathcal{S}(v)} |h_K(u)|^p d\nu(u) \geq 2^{-p} \|x\|^p \nu(\mathcal{S}(v))$$

which implies that  $\|x\| \leq 2\nu(\mathcal{S}(v))^{-1/p} R$ . The quantity  $\nu(\mathcal{S}(v))$  only depends on the dimension  $d$  which completes the proof.  $\square$

## 6 Appendix: Proof of Lemma 3.3

In this section, we provide the proof of Lemma 3.3 which was crucially used in the proof of Theorem 3.1. Before we get to the proof of Theorem 3.1, let us first state and prove a technical result which we then use to prove Theorem 3.1.

**Lemma 6.1.** *Suppose  $f$  is a continuous convex function on  $[0, a]$  with  $f(0) < 0$ . Then for every  $\alpha > 0$  and  $p > 0$ , we have*

$$\int_0^a x^{\alpha-1} |f(x)|^p dx \geq C(\alpha, p) |f(0)|^p a^\alpha$$

where  $C(\alpha, p)$  is the positive constant given by

$$C(\alpha, p) := \inf_{0 \leq \beta \leq 1} \int_0^1 u^{\alpha-1} |u - \beta|^p du. \quad (28)$$

*Proof.* Suppose first that  $f(a) \leq 0$ . By convexity, we have

$$f(x) \leq \frac{x}{a} f(a) + \left(1 - \frac{x}{a}\right) f(0) \leq \left(1 - \frac{x}{a}\right) f(0)$$

and so we have  $|f(x)| \geq (1 - (x/a))|f(0)|$ . As a consequence,

$$\int_0^a x^{\alpha-1} |f(x)|^p dx \geq |f(0)|^p \int_0^a x^{\alpha-1} \left(1 - \frac{x}{a}\right)^p dx. \quad (29)$$

Now let  $f(a) > 0$ . By continuity, there exists  $\beta \in (0, 1)$  with  $f(a\beta) = 0$ . For  $0 \leq x \leq a\beta$ , we have by convexity

$$f(x) \leq \frac{x}{a\beta} f(a\beta) + \left(1 - \frac{x}{a\beta}\right) f(0) = \left(1 - \frac{x}{a\beta}\right) f(0)$$

which implies that

$$\int_0^{a\beta} x^{\alpha-1} |f(x)|^p dx \geq |f(0)|^p \int_0^{a\beta} x^{\alpha-1} \left|1 - \frac{x}{a\beta}\right|^p dx. \quad (30)$$

On the other hand, for  $a\beta \leq x \leq a$ , we have, again by convexity,

$$0 = f(a\beta) \leq \frac{a\beta}{x} f(x) + \left(1 - \frac{a\beta}{x}\right) f(0)$$

which gives

$$\int_{a\beta}^a x^{\alpha-1} |f(x)|^p dx \geq |f(0)|^p \int_{a\beta}^a x^{\alpha-1} \left|1 - \frac{x}{a\beta}\right|^p dx. \quad (31)$$

Combining (31) and (30), we obtain

$$\int_0^a x^{\alpha-1} |f(x)|^p dx \geq |f(0)|^p \int_0^a x^{\alpha-1} \left|1 - \frac{x}{a\beta}\right|^p dx = |f(0)|^p a^{-p} \beta^{-p} \int_0^a x^{\alpha-1} |x - a\beta|^p dx.$$

Because  $\beta < 1$ , we get

$$\int_0^a x^{\alpha-1} |f(x)|^p dx \geq |f(0)|^p a^{-p} \int_0^a x^{\alpha-1} |x - a\beta|^p dx.$$

By the change of variable  $x = au$  and noting that  $0 < \beta < 1$  is arbitrary, we obtain

$$\int_0^a x^{\alpha-1} |f(x)|^p dx \geq |f(0)|^p a^\alpha \inf_{0 \leq \beta \leq 1} \int_0^1 u^{\alpha-1} |u - \beta|^p du.$$

Because of (29), this inequality also holds when  $f(a) \leq 0$ . This completes the proof.  $\square$

We are now ready to prove Lemma 3.3.

*Proof of Lemma 3.3.* It is clear that, without loss of generality, we only need to prove (10) when  $1/2 \leq y_i < 1$  for all  $i = 1, \dots, n$ . The hypothesis on  $\phi$  implies that

$$\int_{y_1}^1 \dots \int_{y_d}^1 |\phi(x)|^p dx_1 \dots dx_n \leq 1.$$

We shall write the integral above in polar coordinates. Let

$$x_1 = y_1 + r \cos \theta_1, \quad x_2 = y_2 + r \sin \theta_1 \cos \theta_2, \dots, \quad x_d = y_d + r \sin \theta_1 \dots \sin \theta_{d-2} \sin \theta_{d-1}.$$

Then

$$\int_{\theta \in \Theta} \int_0^{r_\theta} |g(r, \theta)|^p r^{d-1} \sin^{d-2} \theta_1 \dots \sin \theta_{d-2} dr d\theta_1 \dots d\theta_{d-1} = \int_{y_1}^1 \dots \int_{y_d}^1 |\phi(x)|^p dx \leq 1. \quad (32)$$

for some set  $\Theta$  with

$$g(r, \theta) := \phi(y_1 + r \cos \theta_1, y_2 + r \sin \theta_1 \cos \theta_2, \dots, y_n + r \sin \theta_1 \dots \sin \theta_{d-2} \sin \theta_{d-1})$$

and

$$r_\theta := \min \left( \frac{1 - y_1}{\cos \theta}, \frac{1 - y_2}{\sin \theta_1 \cos \theta_2}, \dots, \frac{1 - y_d}{\sin \theta_1 \dots \sin \theta_{d-2} \sin \theta_{d-1}} \right).$$

Because of the convexity of  $\phi$ , the function  $r \mapsto g(r, \theta)$  is clearly convex on  $(0, r_\theta)$ . Thus by Lemma 6.1, we obtain that for every  $\theta \in \Theta$ ,

$$\int_0^{r_\theta} |g(r, \theta)|^p r^{d-1} dr \geq C(d, p) |g(0, \theta)|^p r_\theta^d = dC(d, p) |\phi(y)|^p \int_0^{r_\theta} r^{d-1} dr$$

where  $C(d, p)$  is defined as in (28). We thus obtain from (32) that

$$1 \geq dC(d, p) |\phi(y)|^p \int_{\theta \in \Theta} \int_0^{r_\theta} r^{d-1} \sin^{d-2} \theta_1 \dots \sin \theta_{d-2} dr d\theta_1 \dots d\theta_{d-1}$$

Converting the above integral back to the regular coordinates, we get

$$1 \geq dC(d, p) |\phi(y)|^p \int_{y_1}^1 \dots \int_{y_n}^1 dy_1 \dots dy_d = dC(d, p) |\phi(y)|^p (1 - y_1) \dots (1 - y_d).$$

This proves (10) with  $c := d^{-1/p} C(d, p)^{-1/p}$ . □

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